# Canonical Perturbation Formulation of Non-linear Dispersion and Chromaticity

Masaru TAKAO

&

J. Schimizu, K. Soutome, and H. Tanaka

JASRI / SPring-8

# Outline

- Objective
- Starting Hamiltonian
- Canonical formulation of non-linear dispersion
- Formula for Momentum compaction
- Comparison of dispersion with experimental data
- Canonical formulation of non-linear chromaticity
- Comparison of chromaticity with experimental data
- Tracking simulation
- Summary

# Objective

- To understand non-linear dynamics:
  - Strong focusing system.
  - Particle dynamics with a large momentum deviation.
  - Isochronous ring.
- To control the higher order terms:
  - Perturbation method has good physical insight.
  - Order by order formula is convenient for the control of higher order terms by the multi-pole magnet.

# Description of System

- Separate function magnets.
- Hard edge magnets.
- No vertical dispersion.
- No skew magnet.
- No solenoid field.
- Normalize momenta and fields by central momentum $p_0$.

# Hamiltonian of Off-Momentum Particle

$$H = -(1 + K_x x)\left(\sqrt{(1+\delta)^2 - p_x^2 - p_y^2} - A_s\right),$$

where

$K_x$ : curvature,

$\delta = (p - p_0)/p_0$ (momentum deviation),

$p_x, p_y$ : normalized momenta by $p_0$,

normalized vector potential :

$$A_s = \frac{1}{2}(1 + K_x x) + \sum_{n=0} \sum_{m=0}^{[n/2]+1} \frac{g_n}{(n+2)!}(-1)^m \binom{n+2}{2m} x^{n+2-2m} y^{2m},$$

$g_0$ : quadrupole field strength, $g_1$ : sextupole, and so on.

# Expanding Hamiltonian by Canonical Variables

$$
\begin{aligned}
H = &\ \text{insignificant constant term} \\
&- \delta K_x x && :\text{linear part} \\
&+ \frac{1}{2(1+\delta)}\left(p_x{}^2 + p_y{}^2\right) + \frac{1}{2}\left(K_x{}^2 + g_0\right)x^2 - g_0 y^2 && :\text{square part} \\
&+ \frac{1}{3!} g_1\left(x^3 - 3xy^2\right) + \frac{1}{2(1+\delta)} K_x x\left(p_x{}^2 + p_y{}^2\right) && :\text{cubic part} \\
&+ \frac{1}{4!} g_2\left(x^4 - 6x^2y^2 + y^4\right) + \frac{1}{8(1+\delta)^3}\left(p_x{}^2 + p_y{}^2\right)^2 && :\text{fourth order part} \\
&+ \text{higher order terms.}
\end{aligned}
$$

Due to the linear part, the central orbit $x = 0$ is no longer the solution of Eq. of motion, which gives the dispersion.

# Transformation to Off-Momentum Trajectory

To eliminate the linear part, we perform the canonical trans-formation shifting the central orbit to the off-momentum trajectory:

$$x = \bar{x} + \delta\eta_1(s), \qquad p_x = \bar{p}_x + \delta\zeta_1(s).$$

Generating function $F_2(x, \bar{p}_x, s)$ of type 2:

$$F_2(x, \bar{p}_x, s) = [x - \delta\eta_1(s)][\bar{p}_x + \delta\zeta_1(s)].$$

New Hamiltonian:

$$\begin{aligned}
\bar{H} &= H + \frac{\partial F_2}{\partial s} \\
&= \text{insignificant constant term} \\
&\quad + \delta\left[\bar{x}\left\{-K_x + \left(K_x^2 + g_0\right)\eta_1 + \zeta_1'\right\} + \bar{p}_x\left(\zeta_1 - \eta_1'\right)\right] \\
&\quad + \delta^2\left[\frac{1}{2}\bar{x}\left(g_1\eta_1^2 + K_x\zeta_1^2\right) + \bar{p}_x(-\zeta_1 + K_x\eta_1\zeta_1)\right] \\
&\quad + \cdots.
\end{aligned}$$

} Linear part

# Dispersion Equation

The condition of eliminating the first order terms in δ of the linear part gives the equation of the first order dispersion:

$$\eta_1' = \zeta_1, \qquad \zeta_1' = -\left(K_x{}^2 + g_0\right)\eta_1 + K_x,$$

i.e.

$$\eta_1'' + \left(K_x{}^2 + g_0\right)\eta_1 = K_x.$$

To eliminate the residual second order terms in δ of the linear part, we perform successive transformation:

$$\bar{x} = \bar{\bar{x}} + \delta^2 \eta_2(s), \qquad \bar{p}_x = \bar{\bar{p}}_x + \delta^2 \zeta_2(s),$$

which gives the equation of the second order dispersion:

$$\eta_2'' + \left(K_x{}^2 + g_0\right)\eta_2 = -K_x + \left(2K_x{}^2 + g_0\right)\eta_1 - \frac{1}{2}\left(2K_x{}^3 + g_1\right)\eta_1{}^2 + \frac{1}{2}K_x \eta_1'^2.$$

# Recurrent Equations for Higher Order Dispersion

Performing the multiply successive transformation:

$$x = \hat{x} + \sum_{n=1} \delta^n \eta_n(s), \qquad p_x = \hat{p}_x + \sum_{n=1} \delta^n \zeta_n(s),$$

we get the recurrent equations for higher order dispersions:

$$\eta_1'' + \left(K_x{}^2 + g_0\right)\eta_1 = K_x,$$

$$\eta_2'' + \left(K_x{}^2 + g_0\right)\eta_2 = -K_x + \left(2K_x{}^2 + g_0\right)\eta_1 - \frac{1}{2}\left(2K_x{}^3 + g_1\right)\eta_1{}^2 + \frac{1}{2}K_x \eta_1'^2$$

$$\vdots$$

$$\eta_n'' + \left(K_x{}^2 + g_0\right)\eta_n = \Omega_n(\eta_1, \cdots, \eta_{n-1})$$

Solving the recurrent equations of the dispersion order by order, we can obtain the higher order dispersions.

# Solving the Dispersion Equation

- Green function method
  - Integrate the formal solution of the recurrent equation:

$$\eta_n(s) = \frac{\sqrt{\beta_x(s)}}{2\sin \pi\nu_x} \oint ds' \Omega_n(s') \beta_x(s') \cos\left[\pi\nu_x - \left|\varphi_x(s) - \varphi_x(s')\right|\right]$$

- Transfer matrix method (we adopted)
  - Regarding $\Omega_n$ as an effective dipole field, we solve the dispersion equation with the periodicity condition by means of the transfer matrix method.

# Formula for Momentum Compaction

Path length of off-momentum trajectory:

$$L(\delta) = \int_0^{L_0} ds \sqrt{\left\{1 + K_x \sum_{n=1} \delta^n \eta_n(s)\right\}^2 + \left\{\sum_{n=1} \delta^n \eta_n{}'(s)\right\}^2}$$

Momentum compaction factor:

$$\frac{L(\delta) - L(0)}{L(0)} \equiv \alpha(\delta) = \sum_{n=1} \delta^n \alpha_n,$$

$$\alpha_1 = \frac{1}{L_0} \int_0^{L_0} ds K_x \eta_1(s),$$

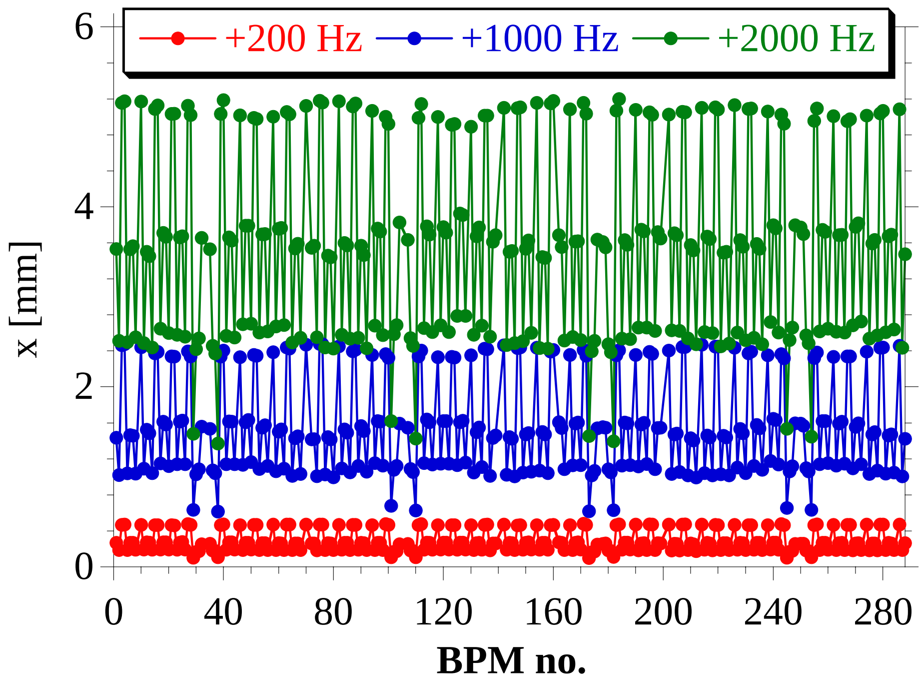
$$\alpha_2 = \frac{1}{L_0} \int_0^{L_0} ds \left[K_x \eta_2(s) + \frac{1}{2} \eta_1{}'^2(s)\right], \cdots$$

Phase slippage factor:

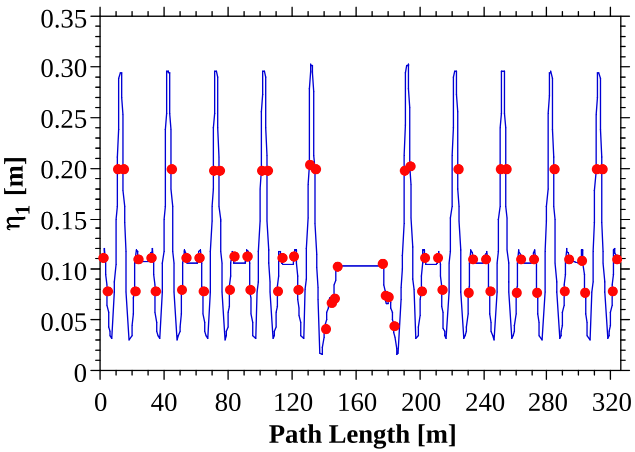
$$\frac{T(\delta) - T(0)}{T(0)} \quad \text{where } T(\delta) = \frac{L(\delta)}{v(\delta)},\ v : \text{velocity.}$$

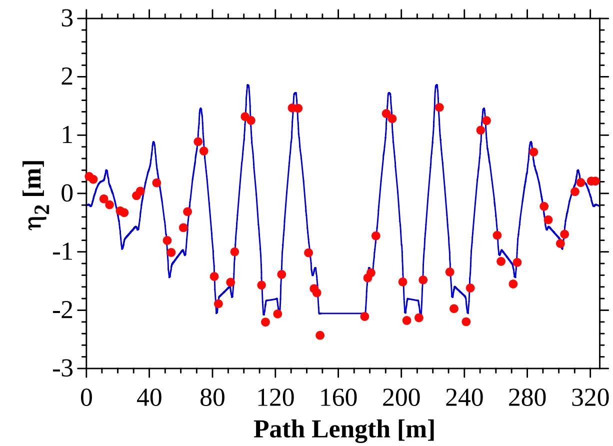
# Dispersion Measurement

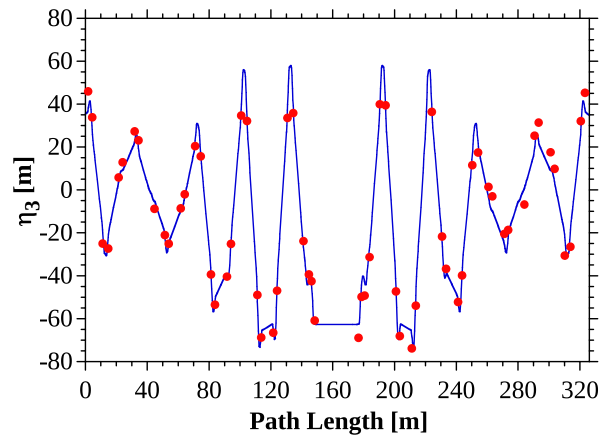
- 288 COD BPM @ SPring-8 storage ring
- Measured range: -1400 Hz ~ + 2600 Hz
- RF-frequency is converted to momentum deviation by using calculated momentum compaction factor.

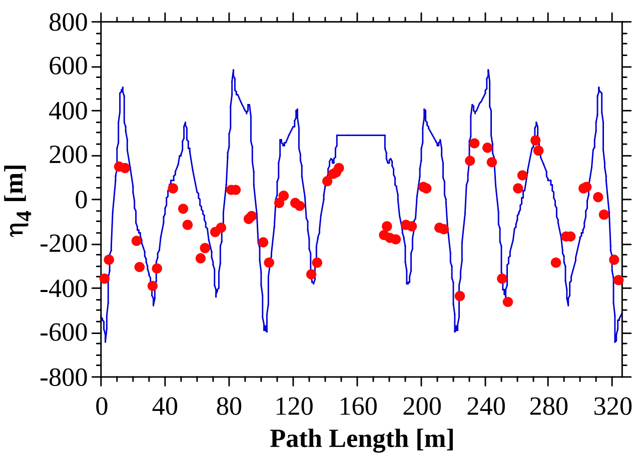


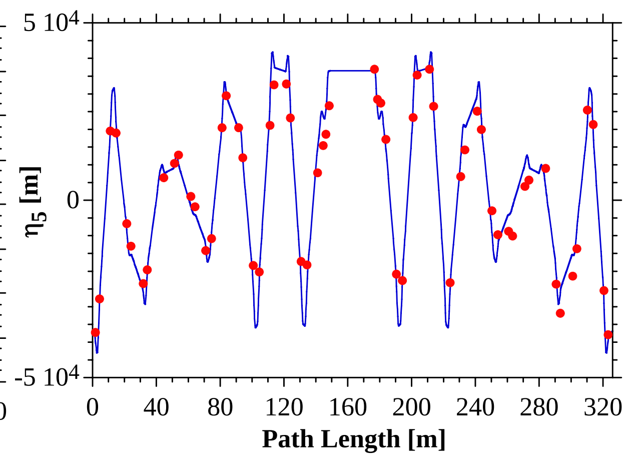
# Dispersion of the SPring-8 Storage Ring



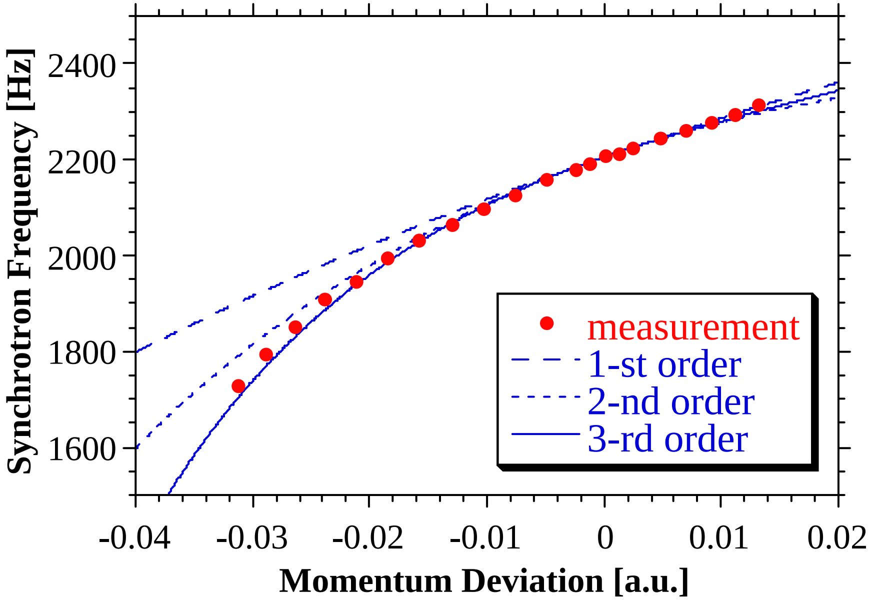



# Synchrotron Frequency

$$f_{\mathrm{sy}} = f_{\mathrm{rf}}(\delta)\sqrt{\frac{\alpha'(\delta)\sqrt{\left(e\hat{V}\right)^2 - U^2(\delta)}}{2\pi h E_{\mathrm{e}}(\delta)}},$$

where

$f_{\mathrm{rf}}$ : rf - frequency, $\hat{V}$ : amplitude of rf - voltage, $U$ : radiation loss per turn,
$h$ : harmonic number, $E_{\mathrm{e}}$ : electron energy.



# Hamiltonian of Off-momentum Betatron Motion

$$H = \frac{1}{2}A_x(\delta)p_x^2 + B_x(\delta)xp_x + \frac{1}{2}C_x(\delta)x^2 + (x \to y),$$

where

$$A_x(\delta) = 1 - \delta(1 - K_x\eta_1) + \delta^2\left(1 - K_x\eta_1 + K_x\eta_2 + \frac{3}{2}\eta_1'^2\right) + \cdots \equiv \sum_{n=0} \delta^n A_{x,n},$$

$$B_x(\delta) = \delta K_x\eta_1' + \delta^2\left(K_x\eta_2' - K_x^2\eta_1\eta_1'\right) + \cdots \equiv \sum_{n=0} \delta^n B_{x,n},$$

$$C_x(\delta) = K_x x^2 + g_0 + \delta g_1\eta_1 + \delta^2\left(g_1\eta_2 + \frac{1}{2}g_2\eta_1^2\right) + \cdots \equiv \sum_{n=0} \delta^n C_{x,n},$$

$$A_y(\delta) = 1 - \delta(1 - K_x\eta_1) + \delta^2\left(1 - K_x\eta_1 + K_x\eta_2 + \frac{1}{2}\eta_1'^2\right) + \cdots \equiv \sum_{n=0} \delta^n A_{y,n},$$

$$B_y(\delta) = 0,$$

$$C_y(\delta) = -g_0 - \delta g_1\eta_1 - \delta^2\left(g_1\eta_2 + \frac{1}{2}g_2\eta_1^2\right) + \cdots \equiv \sum_{n=0} \delta^n C_{y,n}.$$

# Transformation to Hamiltonian of Hill's Equation

To investigate betatron motion, we perform the canonical transformation converting Hamiltonian to that of Hill's equation:

$$z = A_z^{1/2}\bar{z}, \qquad p_z = A_z^{-1/2}\left[\frac{1}{2}A_z^{-1}A_z' - B_z\right]\bar{z} + A_z^{-1/2}\bar{p}_z. \quad (z = x \text{ or } y)$$

Generating function $F_2(z, \bar{p}_z, s)$ of type 2:

$$F_2(z, \bar{p}_z, s) = z\left[\left(\frac{1}{2}A_z^{-1}A_z' - B_z\right)z + A_z^{-1/2}\bar{p}_z\right].$$

New Hamiltonian (of Hill's equation):

$$\bar{H} = H + \frac{\partial F_2}{\partial s} = \frac{1}{2}\bar{p}_x^{\,2} + \frac{1}{2}G_x(\delta)\bar{x}^2 + \frac{1}{2}\bar{p}_y^{\,2} + \frac{1}{2}G_y(\delta)\bar{y}^2,$$

where

$$G_z(\delta) = A_z(\delta)C_z(\delta) + \frac{1}{2}\{\ln A_z(\delta)\}'' - B_z'(\delta) - \left[\frac{1}{2}\{\ln A_z(\delta)\}' - B_z(\delta)\right]^2.$$

# Expansion of Potentials

Potential $G_z(\delta)$ $(z=x,y)$ has the following exansion:

$$G_z(\delta) = \sum_{n=0} \delta^n G_{z,n}.$$

where

$$G_{x,0} = K_x{}^2 + g_0,$$

$$G_{x,1} = -(1 - K_x \eta_1)\left(K_x{}^2 + g_0\right) + g_1 \eta_1 + \tfrac{1}{2}(K_x \eta_1)'' - \left(K_x \eta_1'\right)',$$

$$G_{x,2} = \left[1 - K_x(\eta_1 - \eta_2) + \tfrac{3}{2}\eta_1'^2\right]\left(K_x{}^2 + g_0\right) - (1 - K_x \eta_1) g_1 \eta_1 + g_1 \eta_2 + \tfrac{1}{2} g_2 \eta_1{}^2$$

$$- \tfrac{1}{4}(K_x \eta_1)'^2 + \tfrac{1}{2}\left(K_x \eta_2 + \tfrac{3}{2}\eta_1'^2 - \tfrac{1}{2}K_x{}^2 \eta_1{}^2\right)'' - \left[K_x\left(\eta_2' - K_x \eta_1 \eta_1'\right)\right]',$$

$$G_{y,0} = -g_0, \quad G_{y,1} = -(1 - K_x \eta_1) g_0 - g_1 \eta_1 + \tfrac{1}{2}(K_x \eta_1)'', \quad \cdots.$$

# Action-Angle Variables

Unperturbed Hamiltonian:

$$H_{z,0}(z, p_z, s) = \frac{1}{2} p_z^2 + \frac{1}{2} G_{z,0} z^2 \quad \Rightarrow \quad H_{z,0}(J_z, s) = \frac{J_z}{\beta_z(s)}.$$

$$\Uparrow$$

$$z = \sqrt{J_z \beta_z(s)} \cos\phi_z, \quad p_z = \cdots$$

Here $J_z$ is the action variable of unperturbed system, $\phi_z$ the angle variable, and $\beta_z$ the betatron function.

Equation of motion of unperturbed system and the solution:

$$J_z' = -\frac{\partial H_0}{\partial \phi_z} = 0, \qquad \phi_z' = \frac{\partial H_0}{\partial J_z} = \frac{1}{\beta_z}. \quad \Rightarrow \quad J_z = \text{const.}, \qquad \phi_z(s) = \phi_z(0) + \int_0^s \frac{d\tilde{s}}{\beta_z(\tilde{s})}.$$

# Canonical Perturbation Transformation

Hamiltonian including perturbation potential:

$$H_z(J_z,\phi_z,s) = \frac{J_z}{\beta_z(s)} + V_z(J_z,\phi_z,s),$$

where

$$V_z(J_z,\phi_z,s) = \sum_{n=1} \delta^n V_{z,n}(s) = \frac{1}{2} J_z \sum_{n=1} \delta^n \beta_z(s) G_{z,n}(s)(1+\cos 2\phi_z).$$

Considering the canonical transformation from unperturbed variable ( $\phi_z$, $J_z$ ) to new action-angle variable $(\bar{\phi}_z, \bar{J}_z)$, which is derived by the generating function $S_z$ close to identity transformation:

$$S_z(\phi_z,\bar{J}_z,s) = \phi_z \bar{J}_z + \sum_{n=1} \delta^n S_{z,n}(\phi_z,\bar{J}_z,s),$$

# Canonical Perturbation Procedure 1

New Hamiltonian:

$$\bar{H}(\bar{J},s) = H_0(\bar{J},s) + \sum_{n=1} \delta^n K_n(\bar{J},s),$$

where $K_n$ is the perturbation Hamiltonian:

$$K_1 = V_1 + \boxed{\frac{\partial H_0}{\partial \bar{J}}\frac{\partial S_1}{\partial \phi} + \frac{\partial S_1}{\partial s}},$$

$$K_2 = V_2 + \frac{\partial V_1}{\partial \bar{J}}\frac{\partial S_1}{\partial \phi} + \boxed{\frac{\partial H_0}{\partial \bar{J}}\frac{\partial S_2}{\partial \phi} + \frac{\partial S_2}{\partial s}},$$

$$K_3 = V_3 + \frac{\partial V_2}{\partial \bar{J}}\frac{\partial S_1}{\partial \phi} + \frac{\partial V_1}{\partial \bar{J}}\frac{\partial S_2}{\partial \phi} + \frac{1}{2}\frac{\partial^2 V_1}{\partial \bar{J}\partial \bar{J}}\frac{\partial S_1}{\partial \phi}\frac{\partial S_1}{\partial \phi} + \boxed{\frac{\partial H_0}{\partial \bar{J}}\frac{\partial S_3}{\partial \phi} + \frac{\partial S_3}{\partial s}}, \cdots,$$

$$K_n = F_n(V_1,\cdots,V_n,S_1,\cdots,S_{n-1}) + \boxed{\frac{\partial H_0}{\partial \bar{J}}\frac{\partial S_n}{\partial \phi} + \frac{\partial S_n}{\partial s}}, \cdots.$$

# Canonical Perturbation Procedure 2

The perturbation Hamiltonian $K_n$ naively contains angle variable $\phi$.
Canonical condition ($K_n$ is a function of action variable $J$ only) determines perturbation Hamiltonian $K_n$ and generating function $S_n$:

$$K_n = \langle F_n(V_1,\cdots,V_n,S_1,\cdots,S_{n-1})\rangle = \frac{1}{2\pi}\oint d\phi F_n(V_1,\cdots,V_n,S_1,\cdots,S_{n-1}),$$

$$\frac{\partial H_0}{\partial \bar{J}}\frac{\partial S_n}{\partial \phi}+\frac{\partial S_n}{\partial s} = \boxed{\frac{1}{\beta}\frac{\partial S_n}{\partial \phi}+\frac{\partial S_n}{\partial s} = -\{F_n\}(V_1,\cdots,V_n,S_1,\cdots,S_{n-1}),}$$

where $\{F_n\} = F_n - \langle F_n\rangle$ is oscillating part in $\phi$.

The inhomogeneous term of the deciding equation of $n$-th order generating function $S_n$ consists of lower order generating functions.
The recurrent equation for $S_n$ can be solved order by order.

The perturbation Hamiltonian $K_n$ also contains only lower generating functions, and then can be calculated order by order.

# Canonical Perturbation Procedure 3

e.g. 1-st order

$$K_{z,1}(J_z,s)=\frac{1}{2}\bar{J}_z\beta_z(s)G_{z,1}(s),$$

$$S_{z,1}(\phi_z,J_z,s)=-\frac{1}{4\sin(2\pi\nu_z)}\int_s^{s+L}d\tilde{s}\,\beta_z(\tilde{s})G_{z,1}(\tilde{s})\sin 2[\phi+\psi(\tilde{s})-\psi(s)-\pi\nu_z].$$

e.g. 2-snd order

$$K_{z,2}(J_z,s)=\frac{1}{2}\bar{J}_z\beta_z(s)G_{z,2}(s)$$

$$-\frac{1}{8\sin(2\pi\nu_z)}\bar{J}_z\beta_z(s)G_{z,1}(s)\int_s^{s+L}d\tilde{s}\,\beta_z(\tilde{s})G_{z,1}(\tilde{s})\sin 2[\psi(\tilde{s})-\psi(s)-\pi\nu_z],$$

$$S_{z,2}(\phi_z,J_z,s)=-\frac{1}{4\sin(2\pi\nu_z)}\int_s^{s+L}d\tilde{s}\,\beta_z(\tilde{s})G_{z,2}(\tilde{s})\sin 2[\phi+\psi(\tilde{s})-\psi(s)-\pi\nu_z]$$

$$+\cdots.$$

# Canonical Perturbation Procedure 4

The equation of motion for angle variable in perturbed system:

$$\bar{\phi}_z{}'(s) = \frac{\partial \bar{H}_z}{\partial \bar{J}_z} = \frac{\partial H_{z0}}{\partial \bar{J}_z} + \sum_{n=1} \delta^n \frac{\partial K_{z,n}}{\partial \bar{J}_z}.$$

Higher order chromaticity is the integration of partial derivative of perturbation hamiltonian $K_{z,n}$ over circumference:

$$\xi_{z,n} = \frac{1}{2\pi} \oint ds \frac{\partial K_{z,n}}{\partial \bar{J}_z} (\bar{J}_z, s).$$

e.g. 1-st order

$$\xi_{x,1} = \frac{1}{4\pi} \int_s^{s+L} d\tilde{s}\, \beta_x(\tilde{s}) G_{x,1}(\tilde{s})$$

$$= \frac{-1}{4\pi} \int_s^{s+L} d\tilde{s} \left[ \beta_x \left( K_x{}^2 + g_0 - g_1 \eta_1 \right) + 2\alpha_x K_x \eta_1{}' - \gamma_x K_x \eta_1 \right].$$

# Higher Order Formula for Chromaticity

Repetition of the procedure give the higher order formula for chromaticity.

2-nd order:

$$\xi_{z,2}(s)=$$

$$\frac{1}{4\pi}\begin{pmatrix} \int_s^{s+L} d\tilde{s}\beta_z(\tilde{s})G_{z,2}(\tilde{s}) \\ -\frac{1}{4\sin(2\nu_z)}\int_s^{s+L} d\tilde{s}\beta_z(\tilde{s})G_{z,1}(\tilde{s})\int_s^{\tilde{s}+L} d\tilde{\tilde{s}}\beta_z(\tilde{\tilde{s}})G_{z,1}(\tilde{\tilde{s}})\cos 2\left[\psi_z(\tilde{\tilde{s}})-\psi_z(\tilde{s})-\pi\nu_z\right] \end{pmatrix},$$

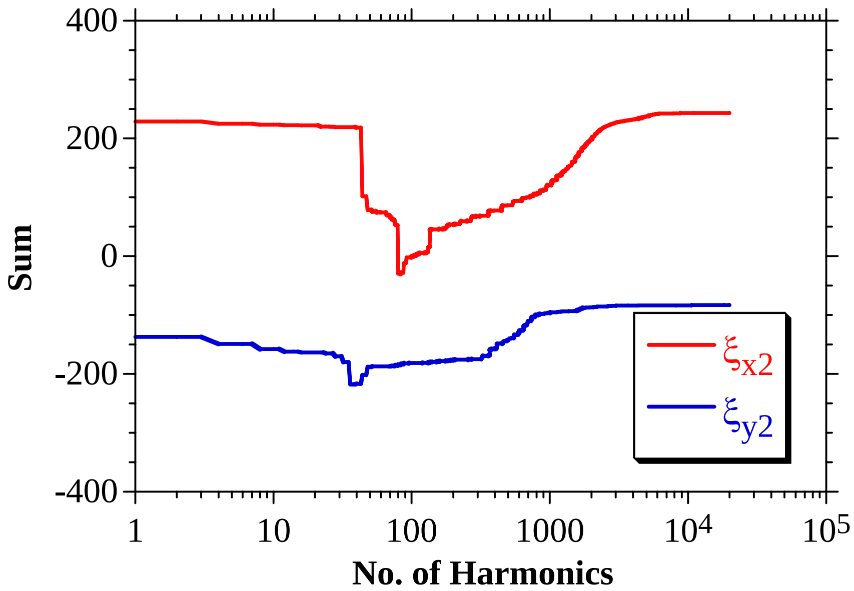
3-rd order:

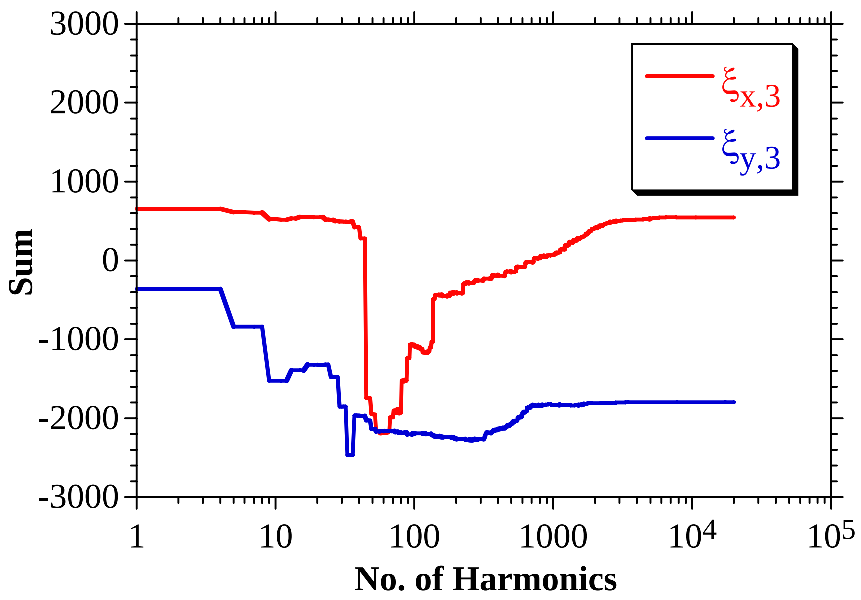
$$\xi_{z,3}(s)=$$

$$\frac{1}{4\pi}\begin{pmatrix} \int_s^{s+L} d\tilde{s}\beta_z(\tilde{s})G_{z,3}(\tilde{s}) \\ -\frac{1}{4\sin(2\nu_z)}\int_s^{s+L} d\tilde{s}\beta_z(\tilde{s})G_{z,2}(\tilde{s})\int_s^{\tilde{s}+L} d\tilde{\tilde{s}}\beta_z(\tilde{\tilde{s}})G_{z,1}(\tilde{\tilde{s}})\cos 2\left[\psi_z(\tilde{\tilde{s}})-\psi_z(\tilde{s})-\pi\nu_z\right] \\ +\cdots \quad (\text{triple integral of } G_{z,1}) \end{pmatrix},$$

# Chromaticity of the SPring-8 Storage Ring 1

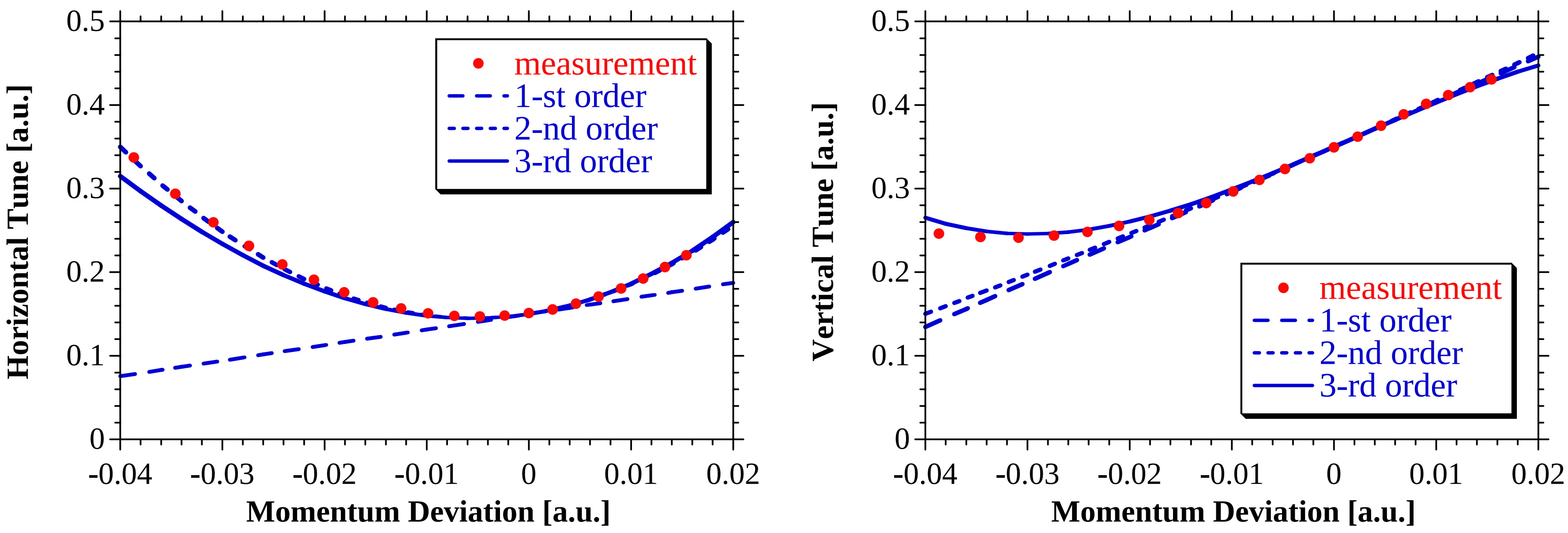
- Applying the formula to the SPring-8 storage ring.
  - $(\nu_x, \nu_y) = (40.15, 18.35), (\xi_x, \xi_y) = (1.8, 5.8)$.
- Integration is carried out by Fourier Transform.
- Division number of element: 100.
- Thickness of sextupole magnet is important for convergence.



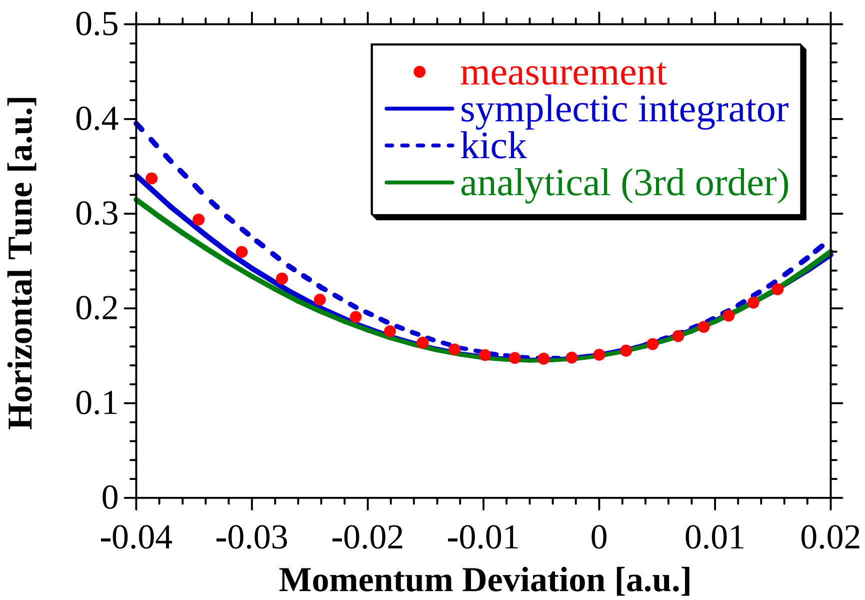
# Chromaticity of the SPring-8 Storage Ring 2

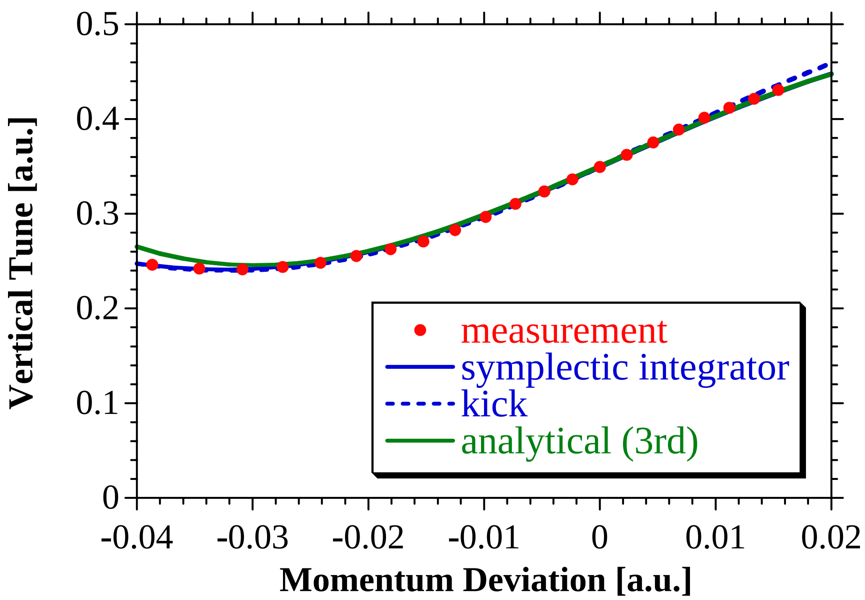
- Comparing the calculation with the measurement of the SPring-8 storage ring.
- Approximation up to 3-rd order is almost sufficient to describe experiment data over measurable range.



# Tracking Simulation

- "RACETRACK" based tracking code.
- Symplectic integration.
- Thickness of sextupole magnet is important for accuracy.



# Summary

- Canonical perturbation formulation for non-linear dispersion and chromaticity was derived.
- Numerical calculation of non-linear chromaticity by tracking code was also done.
- Both well agree with experiments.
- Thickness of sextupole magnet is important to simulate the non-linear dynamics precisely.