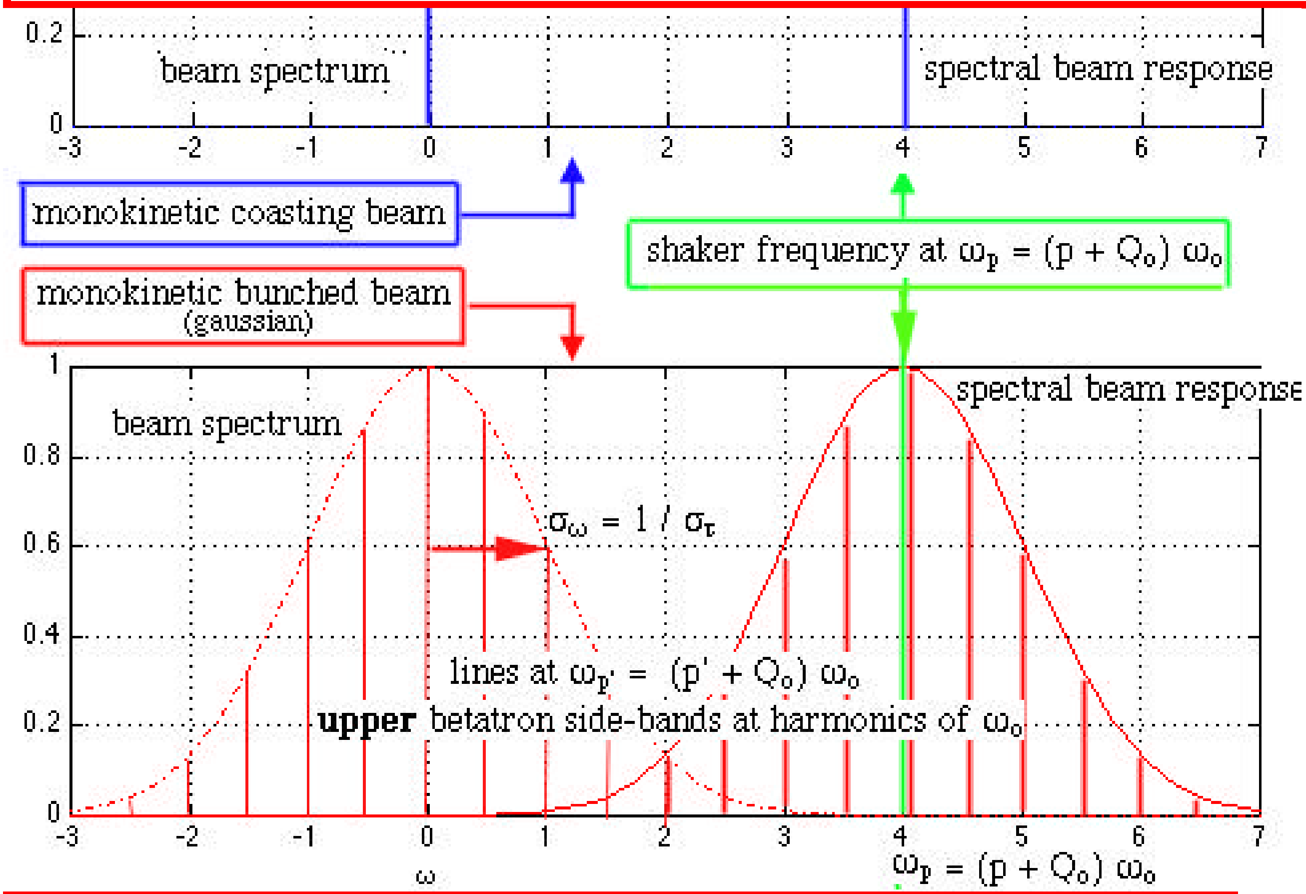
An approximative criterion for the stability of an intense bunched beam at high chromaticity

(G.Besnier, P.Kernel , R.Nagaoka , J.L.Revol)

When the ESRF works with a high chromaticity, the intensity in single bunch mode is suspected to be limited by a fast transverse instability of "post head-tail" type (with growing time shorter than a synchrotron period). Elements of a theory* in frequency space are presented, using J.L. Laclare 's formalism. The work on this topic is part of a thesis by Ph.Kernel : some aspects are recent, not fully achieved and possibly need more reflection.

Look at the spectral response of a monokinetic beam driven by a monochromatic shaker



The spectral response is the beam spectrum shifted to the shaker frequency at ω_p

J.L.Laclare's formalism is more easily understood when we define the matrix elements :

$\mathbf{A}_{\mathbf{p'},\,\mathbf{p}}$ = the amplitude frequency line $\sigma(\omega_{p'})$ in response to a unit excitation at ω_p

= the spectrum of the longitudinal bunch density shifted at ω_p

(later designed as a "shaker mode" spectrum , centered at ω_p)

= $\delta_{p\,,\,p'}$ for a **coasting beam** , = $\exp - [(\omega_p - \omega_{p'})^2\,\sigma_\tau^2 / 2]$ for a **gaussian beam** .

such a matrix **A** works like the"impulse response"of a filter in frequency space

* See also a previous and different analysis by R.D.Ruth and J.M. Wang : Vertical fast blow up in a single bunch . IEEE Transactions on Nuclear Sciences,NS-28,N°3,june 1982.

Transverse modes of a monokinetic bunch, broad band impedance

the stability of a collective bunch oscillation is given by
the imaginary part of the complex frequency shift $\Delta\omega^o_c$ from $Q_o\ \omega_o$

narrow band impedanc $e^*\ Z_T(\omega)$

1)admit the **coasting beam** result :

$$\Delta\omega^o_{cp} = \Lambda\, jZ_T(\omega_p)\ ,\quad \Lambda = \frac{c\ I_o}{4\pi\ Q_o\ E_o/e}.$$

$$[\text{or } \Delta\omega^o_{c\,p'}\ \sigma(\omega_{p'}) = \Lambda\, \delta_{p',p}\ jZ_T(\omega_p)\ \sigma(\omega_p)]$$

2)deduce the **bunched beam** formula :

$$\Delta\omega^o_c\ \sigma(\omega_{p'}) = \Lambda\ \mathbf{A}_{p',p}\ jZ_T(\omega_p)\sigma(\omega_p)$$

(only one input line)

broad band impedance $Z_T(\omega)$

3)same **coasting beam** formula :

$$\Delta\omega^o_{cp} = \Lambda\, jZ_T(\omega_p)\ ,\quad \Lambda = \frac{c\ I_o}{4\pi\ Q_o\ E_o/e}$$

stability : upper side band of

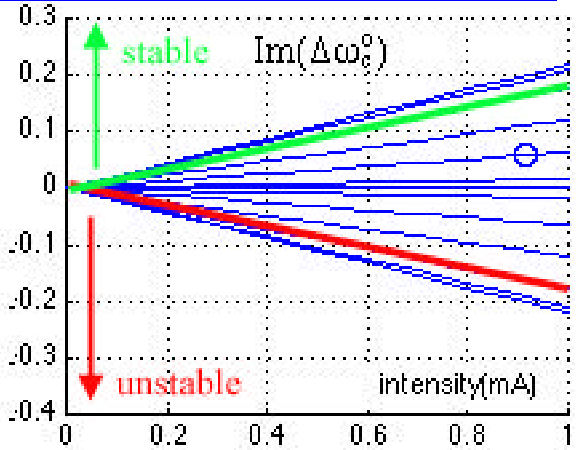
mode "**p**": $\omega^+_{cp} = (\mathbf{p} + Q_o + \Delta\omega^o_{cp})\ \omega_o$

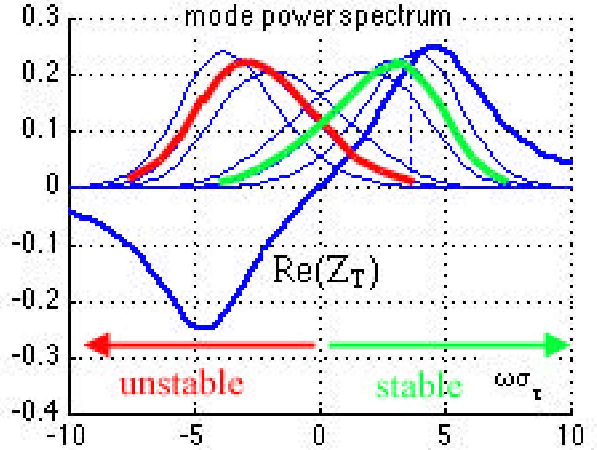
unstable if $\mathbf{Re}\,(\mathbf{Z_T}(\omega^+_{\mathbf{cp}})) < 0$.

4)**bunched beam** : sum all input lines

$$\Delta\omega^o_c\ \sigma(\omega_{p'}) = \Lambda \sum_p jZ_T(\omega_p)\ A_{p',p}\ \sigma(\omega_p)$$

stability :
1- look for **computed solutions**
of this standard eigen-value problem :





stability : 2 - look for **approximative** **solutions** of the eigen-value problem :
$\sigma_q(\omega_p)$ an eigen-mode spectrum associated with an eigen-value $\Delta\omega^o_{c\,q}$,
C_q the associated eigen-value of matrix **A** , the n** :

$$\Delta\omega^o_{c\,q} = \Lambda\, j[Z_{Tq}]_{eff}\ C_q\ ,\ \text{with effective impedance :}\ [Z_{Tq}]_{eff} = \frac{\sum_p Z_T(\omega_p)\ \sigma^2_q\ (\omega_p)}{\sum_p \sigma^2_q\ (\omega_p)}.$$

Approximation of σ_q by a "shaker mode" : $\sigma_q(\omega_p) = \exp -[(\omega_p - \omega_q)^2\ \sigma_\tau^2\ /2]$, then $C_q = \frac{2\sqrt{\pi/3}}{\omega_o\ \sigma_\tau}$ (indpt of q) .

Conclusion : using J.L.Laclare 's formalism, we recover rapidly the following property :

Adding the contributions of upper and lower side-bands : monokinetic coasting and bunched beams are unstable with respect to any impedance which has a resistive component . The frequency shift $\mathrm{Re}(\Delta\omega_{c\ q})$ and growing time $\mathrm{Im}(\Delta\omega_{c\ q})^{-1}$ of any mode "q" can be computed

* by analogy between a shaker tuned at ω_p and a narrow band impedance $Z_T(\omega)$ which overlaps only one line ω_p of the transverse returned signal .

** F.J. Sacherer derived such approximative but very useful formulas for the stability of head-tail sine modes

Bunched beam with energy spread and chromaticity .
Dispersion integral equation .

(G.Besnier, P.Kernel , R.Nagaoka , J.L.Revol)

τ : time delay of a particle , $\tau(t) = \tau_o + \dot{\tau}\, t$

$\dot{\tau}$: a constant of the particule motion during a delay << T_s ," **post head -tail** "regime.

gaussian dispersion function : $g_o(\dot{\tau}) = \dfrac{\exp -[\,\dot{\tau}^2/2\sigma_{\dot{\tau}}^2]}{\sqrt{2\pi}\;\sigma_{\dot{\tau}}}$,

$\sigma_{\dot{\tau}} = \alpha\, \sigma_{E/E_o}$

σ_{E/E_o} the relative energy spread (rms) .

Dispersive effects :

* incoherent betatron frequency spread :

chromatic dispersion of the betatron frequencies , with $\xi = \dfrac{dQ/Q_o}{dE/E_o}$, $\omega_\xi = \dfrac{\xi}{\alpha} Q_o\omega_o$:

$$\omega_\beta(\dot{\tau}) = Q\,\omega_o = Q_o\omega_o + (\,\omega_\xi - Q_o\omega_o)\,\dot{\tau}\ .$$

every line $\omega_p = (p + Q_o)\omega_o$ of the monokinetic bunch is widened and becomes a (narrow) continuous gaussian spectrum : $\sigma(\omega) = \dfrac{1}{|\,\omega_p - \omega_\xi\,|}\, g_o\!\left(\dfrac{\omega - \omega_p}{\omega_p - \omega_\xi}\right)$

with rms width $|\,\omega_p - \omega_\xi\,|\,\sigma_{\dot{\tau}}$.

*A new problem for the stability of a coherent transverse mode :

monokinetic bunched beam:		bunched beam with energy spread:
$\Delta\omega_c^o\,\sigma(\omega_{p'}) = \Lambda \bullet_p\; jZ_T(\omega_p)\; A_{p',p}\;\; \sigma(\omega_p)$	$\rightarrow$	$J_{p'}^{-1}(\Delta\omega_c)\;\sigma(\omega_{p'}) = \Lambda \bullet_p\; jZ_T(\omega_p)\; A_{p',p}\;\; \sigma(\omega_p)$,
(standard algebra)		(non standard)

The (desired) complex frequency shift $\Delta\omega_c$ from $Q_o\,\omega_o$ now is hidden inside the following dispersion integral : $J_p(\Delta\omega_c) = \displaystyle\int \frac{g_o(\dot{\tau})\, d\dot{\tau}}{\Delta\omega_c - (\,\omega_p - \omega_\xi)\dot{\tau}}$

Resolution "by hand" for a mode spectrum $\sigma_q(\omega_p)$, which is centered at a frequency ω_q , leads to a classical : **dispersion integral relation :**

$$J_q^{-1}(\Delta\omega_{cq}) = \Lambda\,[\,j Z_{Tq}]_{eff}\, C_q \quad \text{or} \quad \Delta\omega_{cq}^o\; J_q(\Delta\omega_{cq}) = 1$$

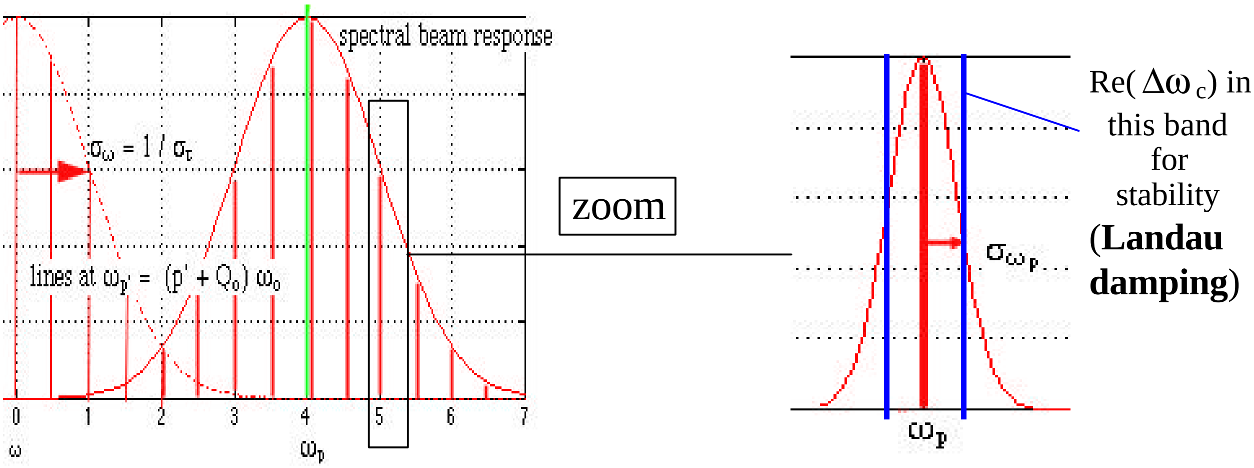
C_q is the associated eigen value of $A_{p,p'}$ and : $[\,Z_{Tq}]_{eff} = \dfrac{\bullet_p\; Z_T(\omega_p)\;\sigma_q^2\,(\omega_p)}{\bullet_p\;\sigma_q^2\,(\omega_p)}$.

relations in complete analogy with the J.L.Laclare* coasting beam result :

$J_p^{-1}(\Delta\omega_{cp}) = \Lambda\, jZ_T(\omega_p)$ or $\Delta\omega_{cp}^o\; J_p(\Delta\omega_{cp}) = 1$, for the mode with p wavelengths

* J.L.Laclare : coasting beam transverse coherent instabilities (ESRF report)

Bunched beam with energy spread and chromaticity .
Stability of a transverse oscillation of post head-tail type .

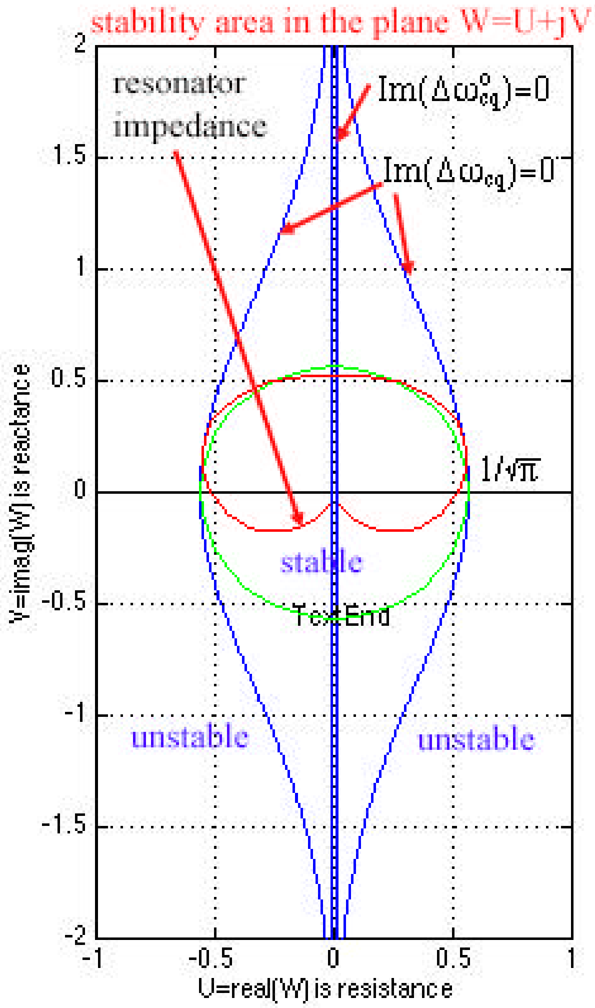


widened side band , rms width $= | \omega_p - \omega_\xi | \sigma_\tau$

stability criterion

Mapping of stability limits $\mathrm{Im}(\Delta\omega^o_{c\,q})=0$ (monocinetic) and $\mathrm{Im}(\Delta\omega_{c\,q})=0$ (energy spread)

on the plane $W = U + jV = \dfrac{\Lambda (Z_{Tq})_{eff} C_q}{\sqrt{2}\, | \omega_\xi - \omega_q | \sigma_\tau} = \dfrac{- j\, \Delta\omega^o_{c\,q}}{\sqrt{2}\, | \omega_\xi - \omega_q | \sigma_\tau}$.



(green) "circle rule" : $|W| < 1 / \sqrt{\pi}$,

or $| \Delta\omega^o_{cq} | < \sqrt{\dfrac{2}{\pi}}\, | \omega_\xi - \omega_q | \sigma_\tau$.

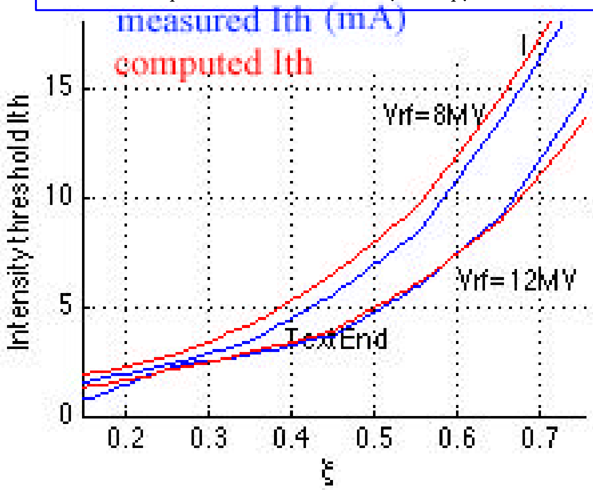
Intensity threshold :

$$I_{th} = \frac{4\, (E_o/e)\;\; \alpha\; (\sigma_E/E)\;\; | \omega_\xi - \omega_q | \sigma_\tau}{\sqrt{2/3}\;\; \beta\, | Z_{Tq} |_{eff}} .$$

ESRF intensity threshold

resonator : βR_s=13.5M Ω, f_r=22GHz,Q=1

with $\omega_q = -\ \omega_r /2$ and $| Z_{Tq} |_{eff} = .6\ \ R_s$:



See a very similar criterion by R.D.Ruth and J.M.Wang (page 1 for reference)